Division

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

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Concrete Pictorial Abstract

referred to as CPA

The concrete-pictorial-abstract approach, based on research by psychologist Jerome Bruner, suggests that there are three steps (or representations) necessary for pupils to develop understanding of a concept. Reinforcement is achieved by going back and forth between these representations.

Concrete representation

The enactive stage - a student is first introduced to an idea or a skill by acting it out with real objects. In division, for example, this might be done by separating apples into groups of red ones and green ones or by sharing 12 biscuits amongst 6 children. This is a 'hands on' component using real objects and it is the foundation for conceptual understanding

Pictorial representation

The iconic stage - a student has sufficiently understood the hands-on experiences performed and can now relate them to representations, such as a diagram or picture of the problem. In the case of a division exercise this could be the action of circling objects

Abstract representation

The symbolic stage - a student is now capable of representing problems by using mathematical notation, for example: 12 ÷ 2 = 6 This is the ultimate mode, for it "is clearly the most mysterious of the three."

Use a range of 'concrete objects' / 'tools to think with'

Begin with real life situations so that children use objects/'tools to think with' to represent the situation. Initially these need to be the objects in the question. Later the objects can be representative.

Using real life questions and modelling using 'concrete objects' as 'tools to think with' will ensure that children develop their ***visualisation skills*** (mentally thinking in your head).

Visualisation is a key intellectual competency—it supports pupil's skills in 'seeing'/visualising what a questions is asking of them. It is very important to develop visualisation so that pupils have this skill as they move from the concrete, to pictorial and then to the abstract.

The development of visualisation skills requires a focussed approach. Visualisation needs to be discussed with the children through the use of questions i.e. What are you picturing in your head?

The importance of visualisation, in terms of how visualisation will support their learning now and in the future, needs to be shared with the pupils.

If, when solving a problem, the focus is on the answer only, then the pupils may not be exposed to the opportunity to develop their ability to visualise, thus restricting their ability to move from concrete representation (enactive) to pictorial representation (iconic) and then to abstract representation (symbolic).

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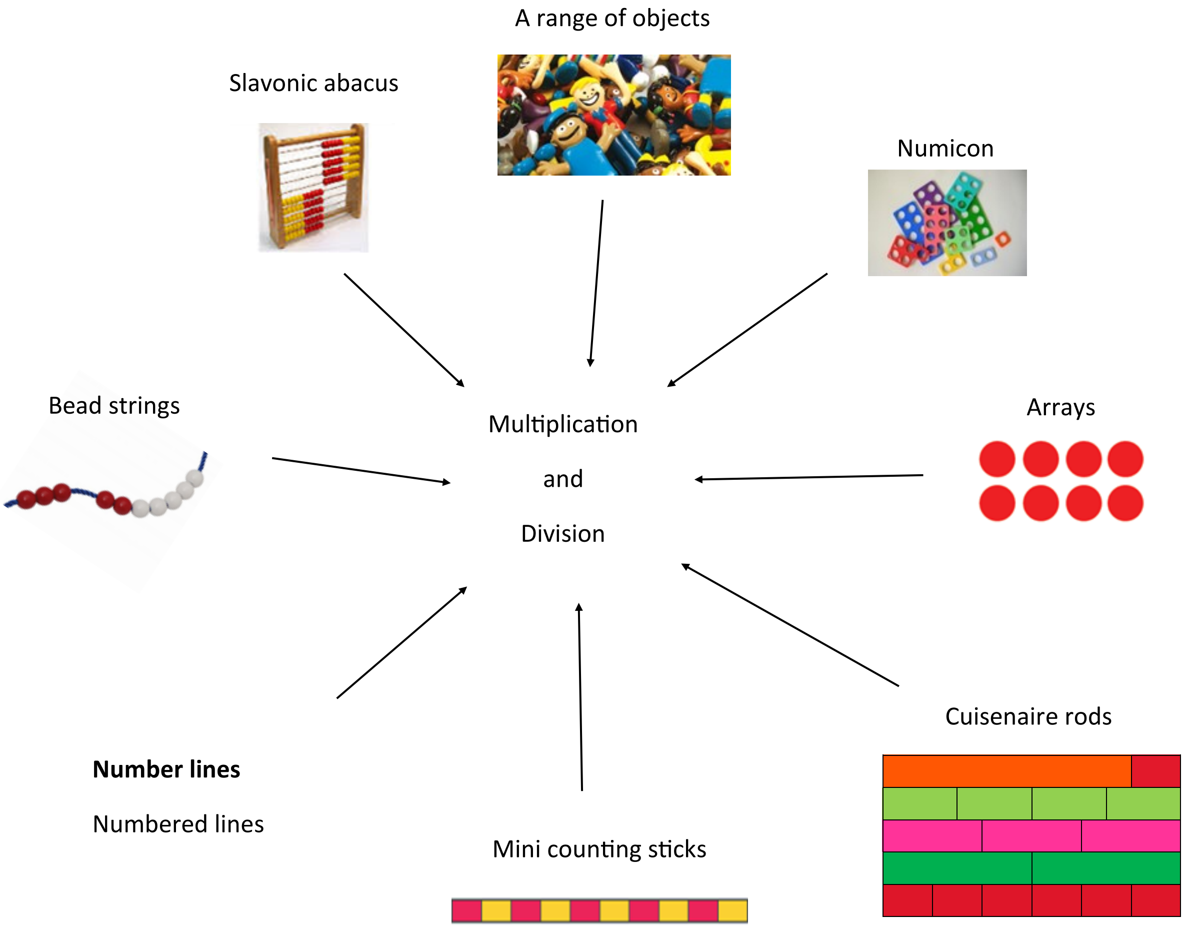
'Tools to think with'



Sharing

Ensure that children have lots of opportunities to experience remainders in context, using real objects in real situations. Children need opportunities to consider what to do with the ones left over after sharing - developing their conceptual understanding. To begin with children just need to offer suggestions for what to do with the ones left over after sharing.

Ensure that children experience remainders:

- where it is appropriate to cut the ones left over after sharing equally - cake, biscuits…
- Where the ones left over are a remainder because we cannot cut the object . For example cars, animals…

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Concrete Pictorial Abstract

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

Set up the classroom so that children can explore sharing

On different tables set up	
Little people into houses (boxes)	Animals into cages (boxes)
Toy cars onto lorries (or into garages)	Toy cars into garages
Fruit into baskets/bowls	Cubes into cups

Sharing - Same dividend - turn over a card to identify the divisor

Children choose a table to work at, the resources they want to work with and move from table to table exploring sharing in different contexts

You will need:

A range of numbers on cards - for example 2, 3, 4, 5… (the divisor).

- The dividend (the amount of objects to be shared remains the same)
- Turn over a card to identify the divisor (how many to share between)
- Share the objects equally between the garages
- Allow children to record in their own way on plain paper.
- Turn over a new card and record in own way...

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Concrete Pictorial Abstract

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

Set up the classroom so that children can explore sharing

On different tables set up	
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Fruit into baskets/bowls	Cubes into cups

Sharing - Same divisor - turn over a card to identify the dividend

Children choose a table to work at (the resources they want to work with) and move from table to table exploring sharing in different contexts

You will need:

A range of numbers on cards (the dividend), for example 8, 9, 10, 12, 15, 16, 18, 20, 21, 24, 25, 28

- Turn over a card to identify the amount of objects needed (the dividend). Count them out or use your subitising skills.
- The divisor (the number to share the objects between stays the same
- Allow children to record in their own way on plain paper
- Turn over a new card (the divisor) and record in own way…

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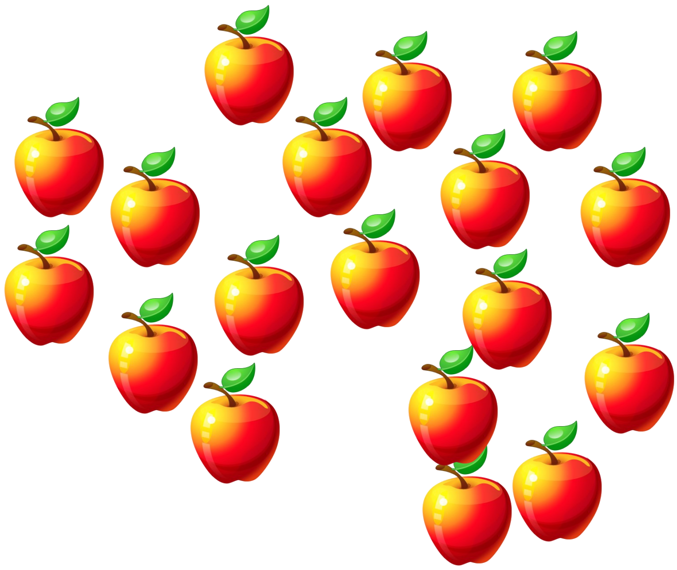
Concrete Pictorial Abstract

Grouping

Make links to subitising

There are 18 apples in a box. We need to put 3 apples into each bag. How many bags will we need?



Concrete Pictorial Abstract

Take out groups of 3 apples

Use **subitising** to identify three (not counting).

Put the three apples into a bag

Ensure that the number of groups (the answer to the question) is put back into the context of the question. *We need 3 bags.*

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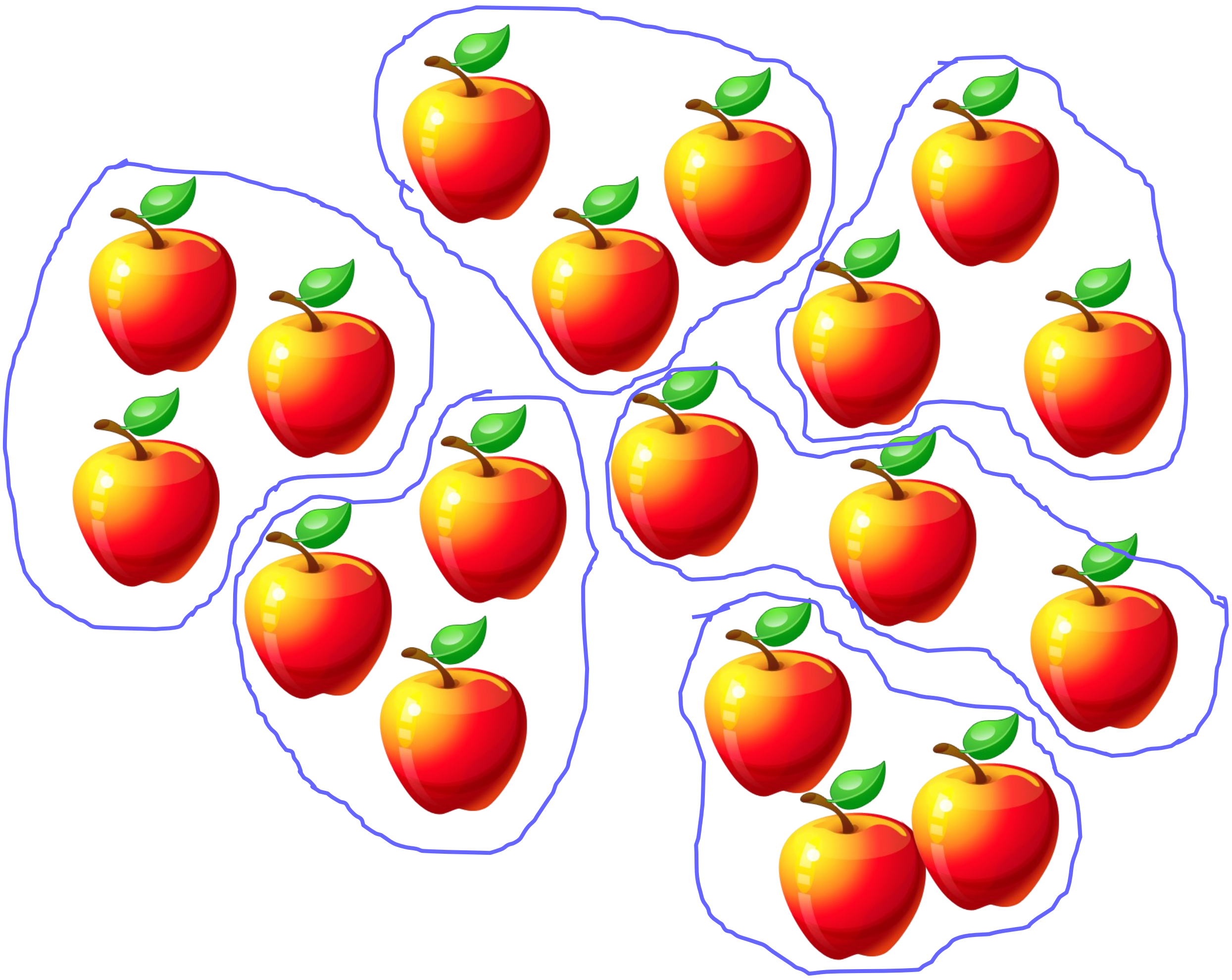
Concrete Pictorial Abstract

There are 18 apples in a box. We need to put 3 apples into each bag. How many bags will we need?



Ensure that the annotation on the image is discussed in the context of the question So, for example, what does the set represent? (in this context - it is the bag)

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Modelling grouping

Is this a Concrete representation?

Give the problem orally - told as a story rather than reading it out.

24 of these new cars at the factory need to go to the garage.

I have to put them on to car transporters to take them to the garage.

Each car transporter can only hold 4 cars.

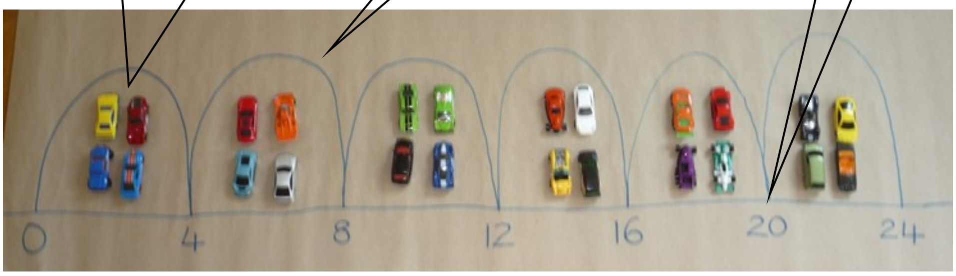
How many car transporters do I need to book?

Using 'real' objects—Concrete representations

1. Ask the children to find groups of 4 from the set of 24 cars using their subitising skills (not counting) and to arrange them on the paper so that we can **see** that there are 4 in each group without counting.

2. Model drawing the jumps on the number line.

3. Model writing the numbers on the number line.



Ask questions like...

What if we got a phone call saying the garage now needs 32 cars?

Or 20 cars?

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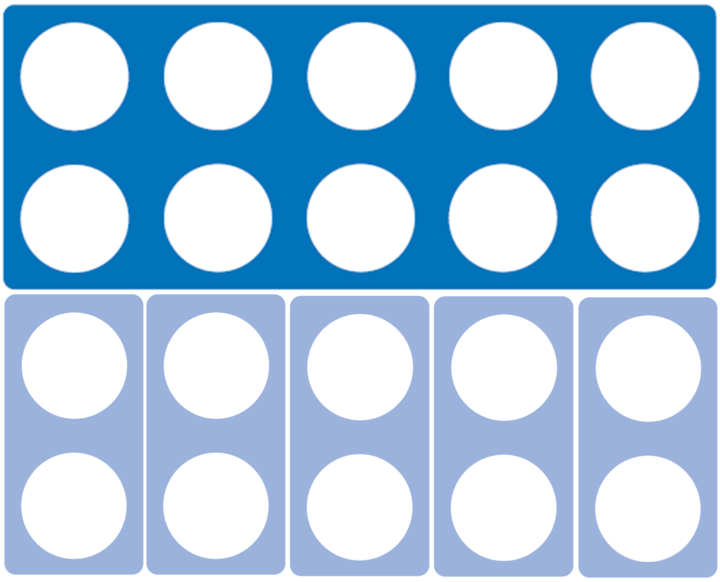
Exploring Numicon

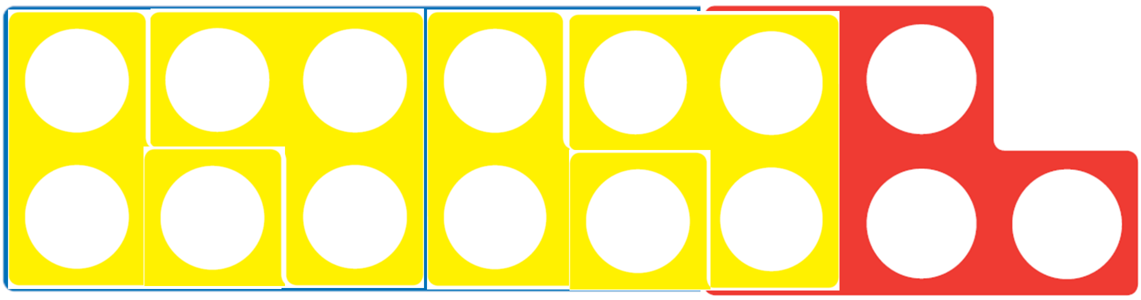
Use Numicon to find out how many twos are the same as 10.

Overlay the Numicon two plates on top of the Numicon ten plate.



Use Numicon to find out how many threes are the same as 15

Overlay the Numicon three plates on top of the 15.



Explore with other numbers

Ensure that children experience remainders

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Grouping - finding out how times the divisor goes into the dividend

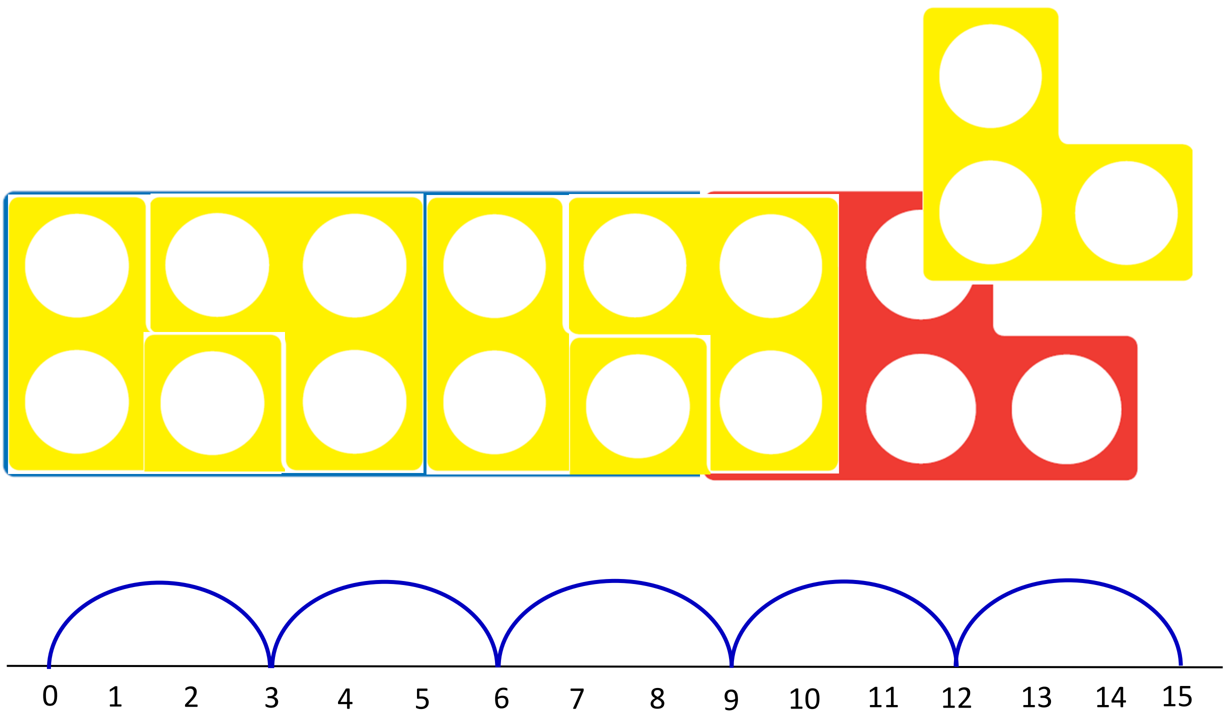
Using Numicon and a numbered number line

Grouping

Use Numicon to find out how many threes are the same as 15—at this stage the division symbol does not need to be introduced

Overlay the Numicon three plates on top of the 15.

Represent this on a numbered number line.



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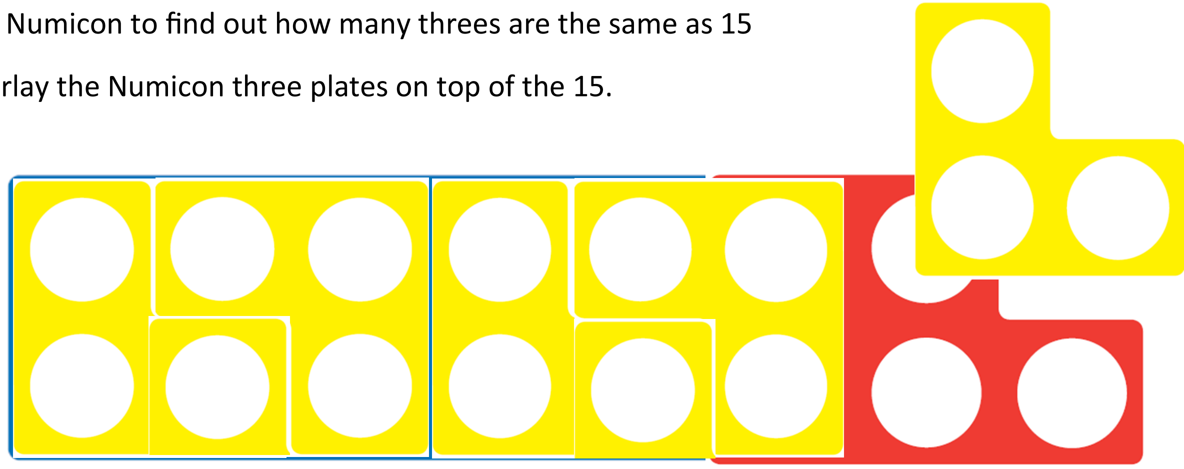
Grouping

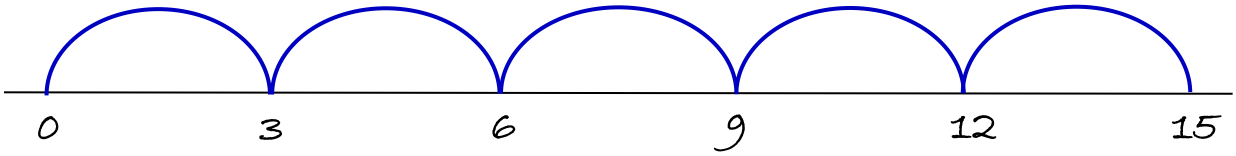
Numicon and the empty number line

Grouping - finding out how times the divisor goes into the dividend

Use Numicon to find out how many threes are the same as 15

Overlay the Numicon three plates on top of the 15.

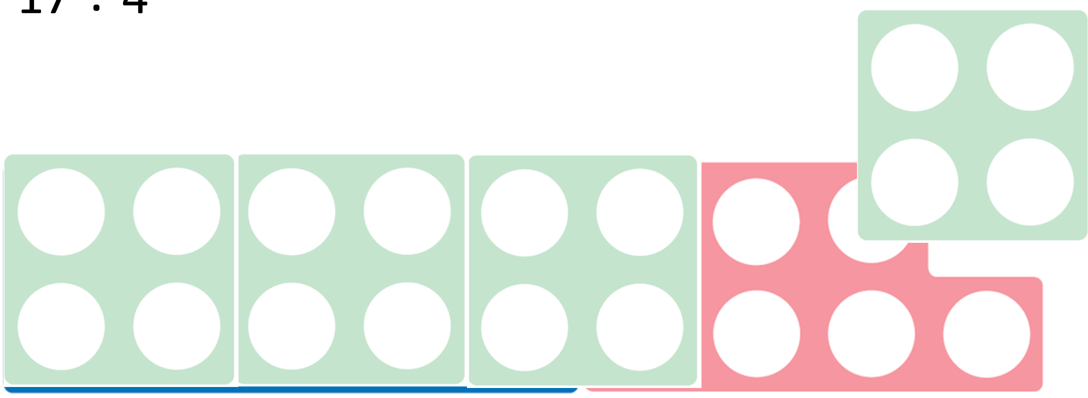


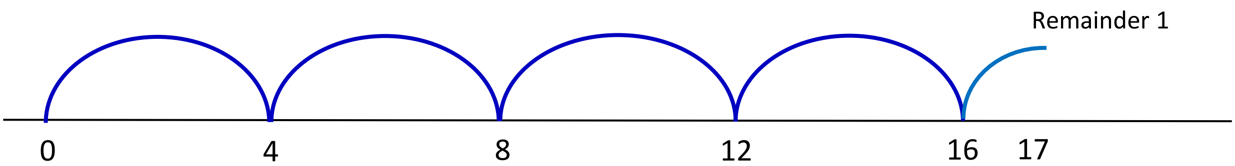


Grouping

Exploring remainders with Numicon

17 ÷ 4





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Grouping - using Cuisenaire rods

Grouping - finding out how times the divisor goes into the dividend

Make 12 using Cuisenaire rods - a ten rod and a two rod

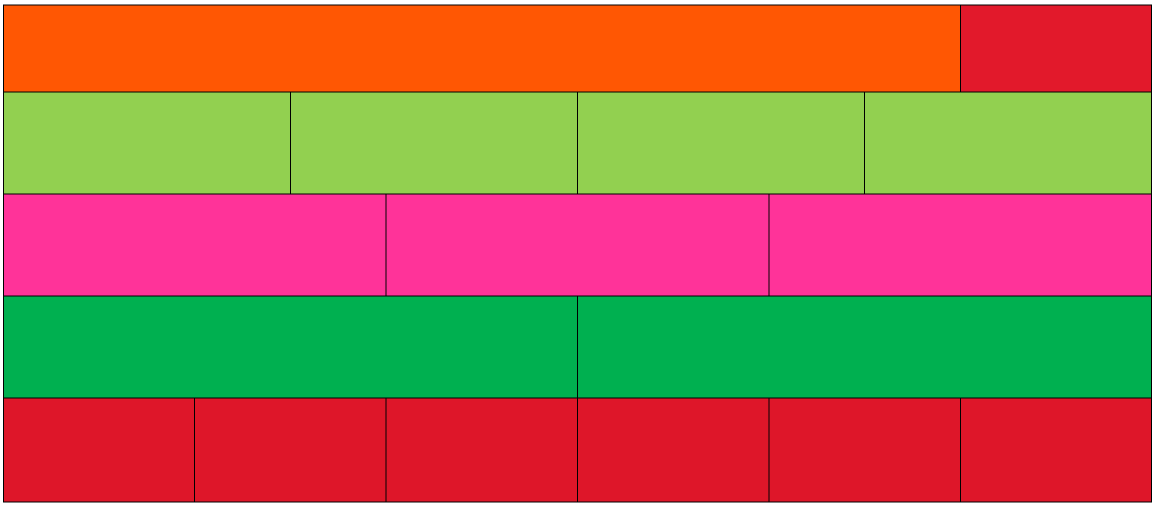
Use Cuisenaire to find out how many threes are the same as 12

Lay the Cuisenaire 'three 'rods below the 12

Explore what other rods will divide into 12 exactly?

Explore with other dividends - no remainder

Write statements to go with the representation.



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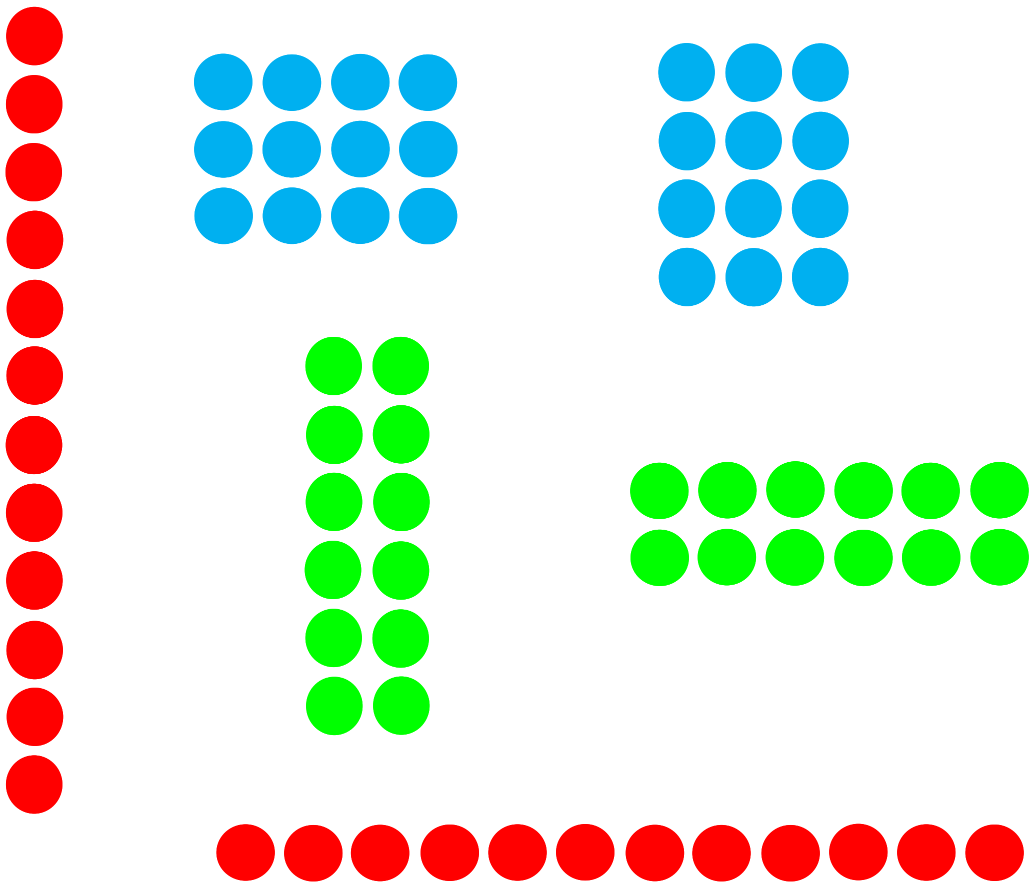
Arrays

Here are 12 counters.

Arrange them in equal rows.

Try different ways



Investigate arrays

Explore making arrays with different numbers

Here are 6, 9, 12, 13, 15, 16, 17, 18, 20, 24, 25, 36... counters

Find different ways to arrange them in equal rows.

What do you notice? What's the same? What's different?

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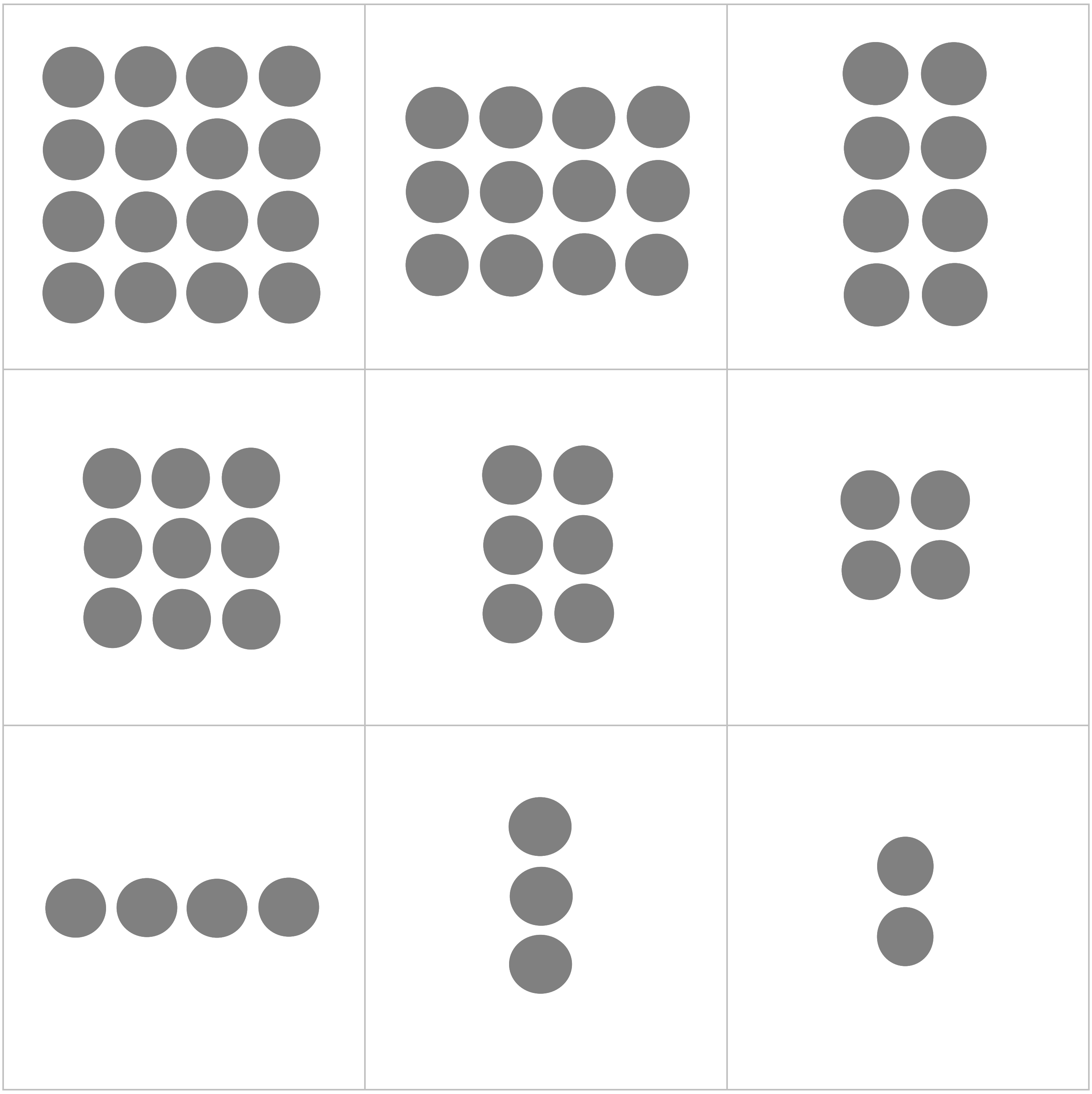
Array cards

Give children lots of opportunities to make and explore arrays: through making them with counters, by printing with circular objects or by pressing circular objects into play-dough.

Use the array cards below as flash cards. Ensure that the children can already subitise.

Let the children see the card for a short time and ask then to visualise the pattern of dots and say how many. Play snap. Play pairs.

To begin with children will need to count the number of dots but with practise they will learn to recognise the pattern of dots.



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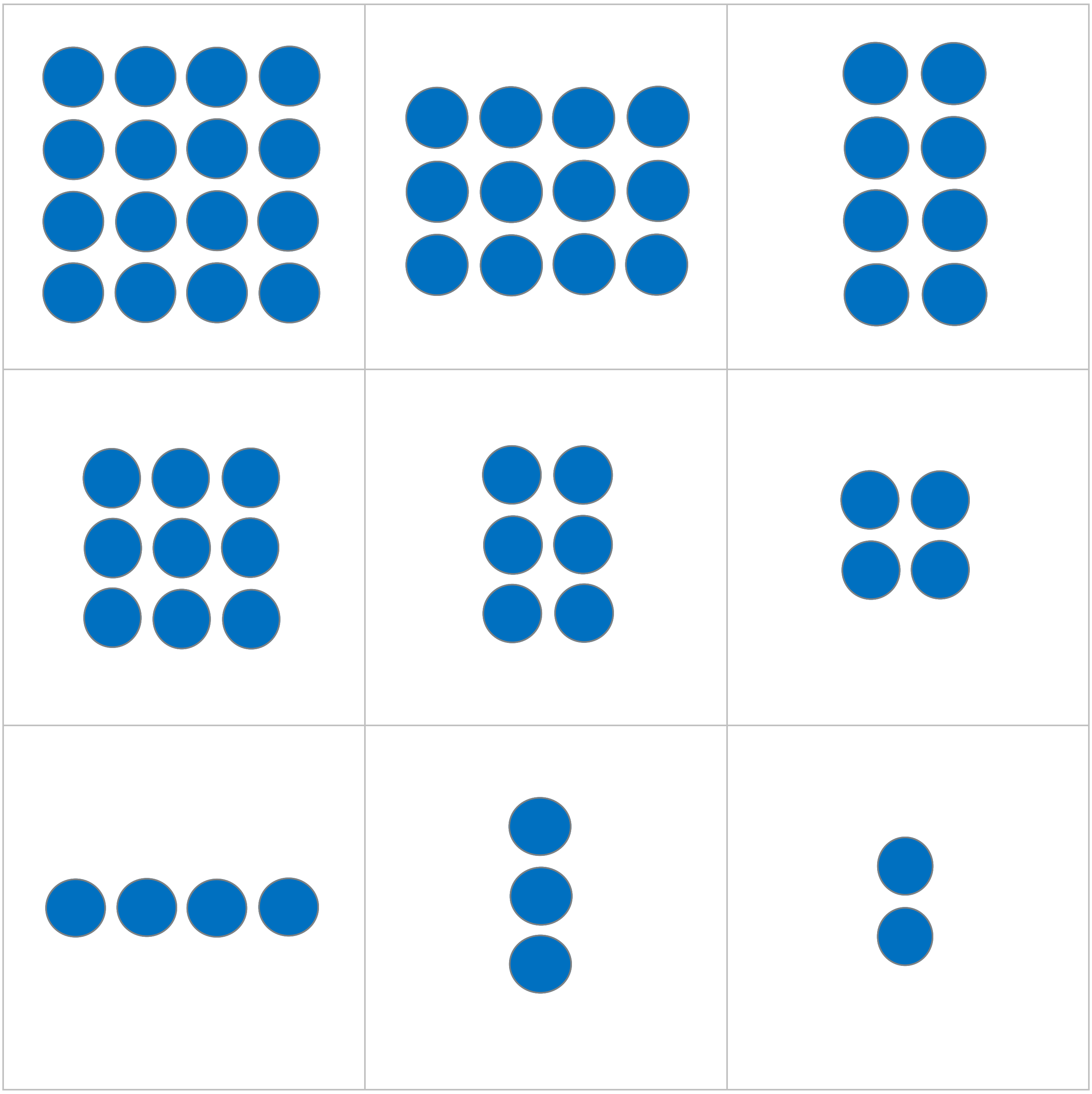
Print onto yellow paper for children who experience difficulties with the contract of back dots on white paper.



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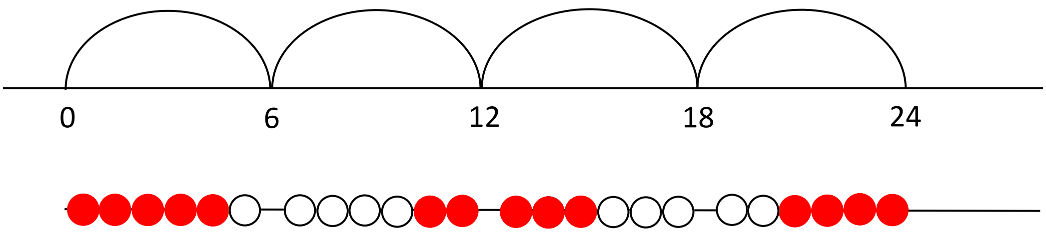
Additive chunking / ‘Chunking up’

24 ÷ 6

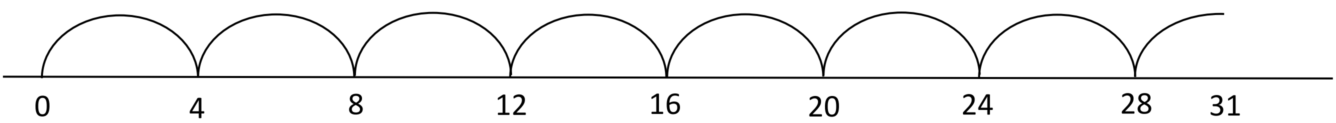


24 ÷ 6 = 4

Discuss with the children how each of the numbers in the calculation relates to the model. The 24 is the number to be divided (the dividend), the 6 is the size of each chunk (the divisor) and the 4 is the number of chunks of 6 in 24 (the quotient).

Additive chunking / ‘Chunking up’ with remainders

31 ÷ 4



31 ÷ 4 = 7 remainder 3

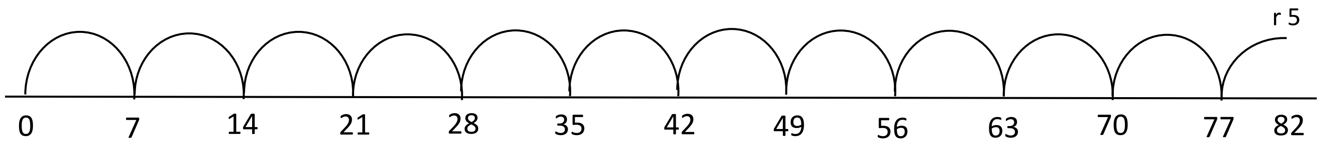
31 ÷ 4 = 7 r 3

Additive chunking / 'Chunking up' with remainders

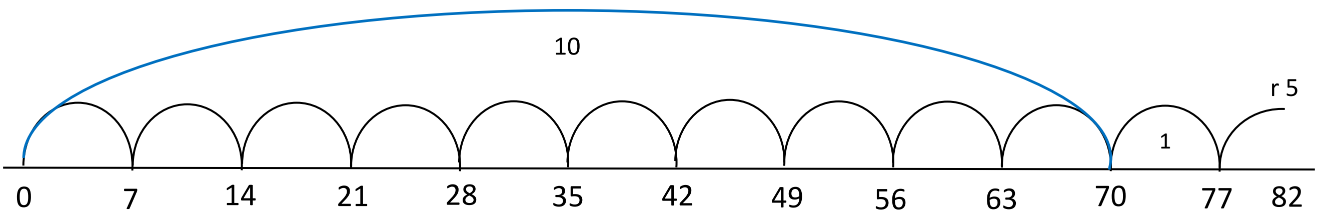
Moving towards efficiency

"The challenge in division is to keep track of 3 different sets of working numbers simultaneously: the chunks needed to make the calculation efficient, the number of groups i.e. the quotient and finally how much of the dividend has been used/is left. All 3 sets of numbers can be recorded simultaneously on an empty number line, the chunks taking the form of 'jumps'." (Bradford, 2011)

82 ÷ 7



Children annotate their chunking using a coloured pencil, to show more efficient chunking based on known facts.



82 ÷ 7 = 11 remainder 5

or

82 ÷ 7 = 11 r 5

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Create and begin to explore your first fact box

Initially, ensure the divisor relates to multiplication facts that children **know and can use fluently**.

In Year 2 children are learning the multiplication and division facts for the 2, 5 and 10 multiplication tables and are developing their fluency in applying these. They are also learning to count in steps of 2, 3 and 5 from 0 and in 10s from any number

In Year 3 children are learning the multiplication and division facts for the 3, 4 and 8 multiplication tables - therefore they should be fluent with the 2, 5 and 10 multiplication tables.

In Year 4 children are learning the multiplication and division facts for the 6, 7 and 9, 11 and 12 multiplication table - therefore they should be fluent with the 2, 3, 4, 5, 8 and 10 multiplication tables

Important

Children need to have developed fluency in the doubling and halving of odd and even numbers in order to use fact boxes effectively.

It is important that children have developed fluency or have fluent mental strategies for:

Doubling odd one-digit and two-digit numbers - 7, 9, 11, 13, 15,17, 19, 21, 23, 25, 27,29...

Halving even two-digit numbers where the answer is an odd number - 14, 18, 22, 26, 30...

$1 \longrightarrow 5$

$2 \longrightarrow 10$

$4 \longrightarrow 20$

$10 \longrightarrow 50$

$5 \longrightarrow 25$

$20 \longrightarrow 100$

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Create and begin to explore your first fact box

Year 2 example:

Create a fact box with a group of children

Use a multiplication table that the children know fluently.

Create the fact box with the children, using this format.

Write the facts one at a time.

Children are usually happy with one lot of 5 and two lots of 5 but when the teacher writes 4 lots of 5 next, they often query this and ask why it is not three lots that go next. At this point it is important to talk to the children about the difference between what they have already experienced, that is 'times tables' which list all the multiples of 5 and a fact box which lists some of the multiples of 5.

Ask the children if it is possible to use the fact box to calculate 3 lots of 5 (children will know what 3 lots of 5 are but, we are asking them to identify how they can derive this fact from the information in the fact box). This can cause 'cognitive conflict' for children who have learnt the multiplication facts for 5 by rote. Here we are asking children to apply what they know in a more complex way. We are asking them to notice a connection - to notice that 2 lots of 5 added to 1 lot of 5 is equal to 3 lots of 5. Representing this as a array would be useful. However, if children cannot, fairly quickly, make this connection, then they are not ready to explore fact boxes.

Children will know the product of 4 lots of 5 but, we also ask them how they might use another fact in the box - double 2 lots if 5.

By this stage the children will have taken on board that a fact box includes some of the multiples of 5 and therefore will be happy that the next two multiples are 10 lots of 5 and 5 lots of 5. Again it is important to ask the children to notice connections between these two facts and other facts in the fact box.

Ask children to write out this fact box for themselves - use A3 paper and felt pens.

Ask the children to use the fact box to find the product of 7 lots of 5.

Once children are confident, ask them to work independently to find 9 lots of 5, 11 lots of 5, 13 lots of 5, 19 lots of 5 …

On the next page is an example of Jens's work. Notice his effective use of colour.

$1 \longrightarrow 5$

$2 \longrightarrow 10$

$4 \longrightarrow 20$

$10 \longrightarrow 50$

$5 \longrightarrow 25$

$20 \longrightarrow 100$

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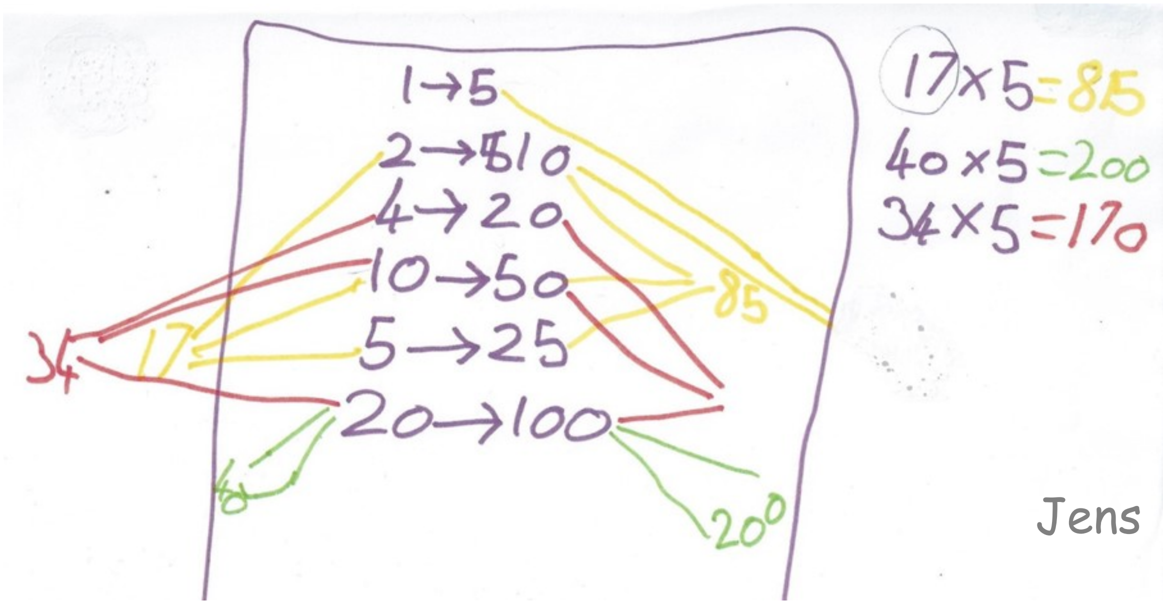
Begin to use your first fact box for multiplication

Here is an example of Jens's work - Year 2

Notice his effective use of colour.



1 ⟶ 5

2 ⟶ 10

4 ⟶ 20

10 ⟶ 50

5 ⟶ 25

20 ⟶ 100

Spend time exploring fact boxes, developing children's 'number sense' and making the links to known facts (times tables), prior to calculating the answers to questions.

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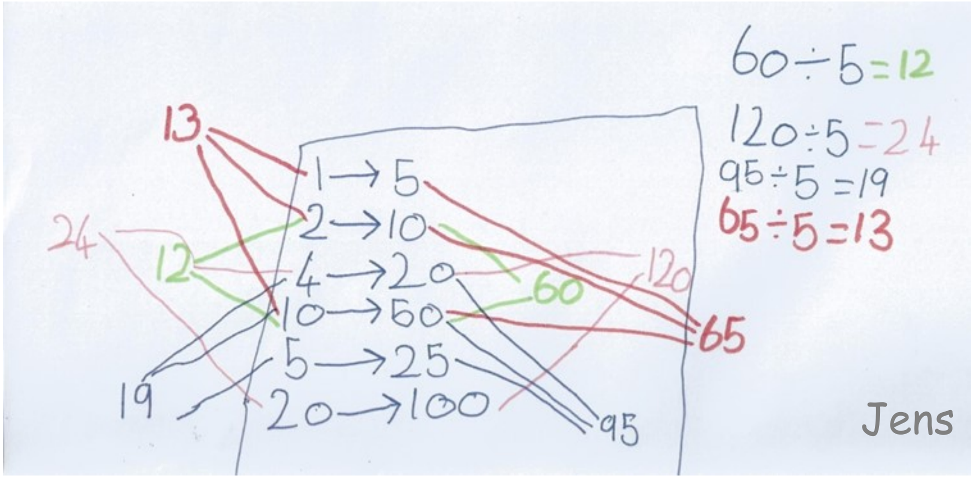
Begin to use your first fact box for division

Here is an example of Jens's work - Year 2

Notice his effective use of colour.



1 → 5

2 → 10

4 → 20

10 → 50

5 → 25

20 → 100

- Create fact boxes from a given divisor. Identify questions that you can answer using your fact box.
- **Spend time exploring fact boxes, developing children's 'number sense' and making the links to known facts (times tables), prior to calculating the answers to questions.**
- Year 2 - use 2, 3, 5 and 10 as the divisor, making links to multiplication facts
- Year 3 - use 2, 3, 4, 5 and 8 as the divisor, making links to multiplication facts
- Year 4 - include 6, 7, 9, 11 and 12 as the divisor, making links to multiplication facts.

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Exploring fact boxes

Create a fact box / 'partial multiple table' for 8 (Thompson, 2005)

Begin with smaller dividends and smaller divisors

A fact box (partial multiples of the divisor)

1	→	8
2	→	16
4	→	32
10	→	80
5	→	40
20	→	160

Children need to develop their confidence in *larger multiples* of *one-digit* numbers. They use the basic skills of doubling and halving and multiplying by 10 to create a fact box or 'partial multiple table' (Thompson, 2012 p.46).

"*Three, twelve or thirty 8s* can be found by addition and if the dividend is a larger number, the table can easily be extended.

The fact box should be used get a 'feel' for the size of the answer - (estimation) through addition.

Adding twenty multiples and five multiples of 8 will give a total of 200 (**too small)**

Adding twenty multiples and ten multiples of 8 will give a total of 240 (t**oo big**)

Therefore the quotient will be more than 25 but less than 30.

- Use pre-prepared fact boxes to explore what can be calculated using the facts.
- Create fact boxes from a given divisor. Identify questions that you can answer using your fact box.
- **Spend time exploring fact boxes, developing children's 'number sense' and making the links to known facts (times tables), prior to calculating the answers to questions.**
- Year 3 - use 2, 3, 4, 5, 8 and 10 as the divisor, making links to multiplication facts
- Year 4 - include 6, 7, 9, 11 and 12 as the divisor, making links to multiplication facts.

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Using this fact box - how would you calculate the following?

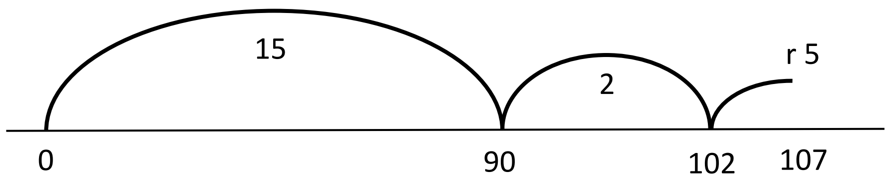
Moving towards efficiency

Jumps are representative

107 ÷ 6

A fact box (partial multiples of the divisor)	
1 →	6
2 →	12
4 →	24
10 →	60
5 →	30
20 →	120

Bank
15
2
r 5



107 ÷ 6 = 17 remainder 5

Using the facts in the fact box

How would you *efficiently* calculate the following?

151 ÷ 6

86 ÷ 6

192 ÷ 6

205 ÷ 6

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Using this fact box - how would you calculate the following?

Moving towards efficiency

Jumps are representative

$82 \div 7$

A fact box (partial multiples of the divisor)

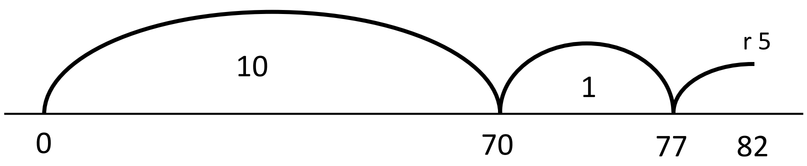
1	→	7
2	→	14
4	→	28
10	→	70

Bank

10

1

r 5



Exploring the fact box

Ask children to try other calculations where the divisor is 7, so that children become familiar with the multiples of 7, they make links to their multiplication facts and can focus on the calculation without, at this stage, having to create a new fact box for each calculation.

Ask children to write other questions they could calculate using their fact box - they can extend their fact box.

Write questions without remainders.

Write questions with remainders.

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Children choose their starting point

Noticing and making connections

Fix the divisor so that children can make connections

Building confidence…	Something to make you think…	Challenge yourself…
$72 \div 4$	$127 \div 6$	$93 \div 7$
$104 \div 4$	$84 \div 6$	$201 \div 7$
$81 \div 4$	$191 \div 6$	$167 \div 7$
$144 \div 4$	$164 \div 6$	$206 \div 7$

Reveal the questions below after the first four quotients have been calculated

Use adjustment to find the quotient to the questions below.

$208 \div 4$	$166 \div 6$	$164 \div 7$
$83 \div 4$	$168 \div 6$	$\square \div 7 = 23$ r 1

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What do you notice?

Making connections

Building confidence…	Something to make you think…	Challenge yourself…
92 ÷ 4	208 ÷ 8	112 ÷ 16
102 ÷ 6	171 ÷ 9	221 ÷ 13
81 ÷ 3	204 ÷ 12	391 ÷ 17
112 ÷ 7	224 ÷ 14	468 ÷ 26

When you have found the quotient for these calculations and you have identified connections within a group of questions and between groups of questions, add two more questions in one of the sets - so that adjustment can be used to find the quotient.

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Exploring remainders

Using knowledge of 2, 3, 4, 5, 8 and 10 multiplication tables and where needed ‘chunking up’ on a number line, sort these calculations into those that **have** a remainder and those that **do not have** a remainder.

Explain your reasoning.

17 ÷ 2	72 ÷ 5	80 ÷ 5	54 ÷ 3
23 ÷ 3	47 ÷ 5	55 ÷ 5	27 ÷ 3
11 ÷ 2	93 ÷ 10	70 ÷ 5	51 ÷ 3
23 ÷ 2	46 ÷ 5	32 ÷ 8	98 ÷ 2
91 ÷ 2	84 ÷ 10	80 ÷ 8	54 ÷ 2
19 ÷ 8	71 ÷ 5	64 ÷ 8	78 ÷ 2
31 ÷ 8	56 ÷ 10	24 ÷ 8	36 ÷ 2
39 ÷ 8	56 ÷ 5	90 ÷ 10	88 ÷ 4
15 ÷ 8	27 ÷ 4	70 ÷ 10	24 ÷ 4
36 ÷ 8	83 ÷ 4	120 ÷ 10	36 ÷ 4
20 ÷ 8	22 ÷ 4	60 ÷ 10	28 ÷ 4
25 ÷ 3	54 ÷ 4	24 ÷ 3	95 ÷ 5

In year 3 devise questions using the following divisors: 2, 3, 4, 5, 8 and 10

In year 4 devise questions using the following divisors: all of the above plus 6, 7, 9, 11, and 12

See Appendix for these cards

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Sets of questions to develop reasoning about division

$32 \div 4$
$32 \div 2$
$32 \div 8$

$12 \div 3$
$12 \div 6$
$24 \div 6$

$25 \div 5$
$50 \div 5$

$32 \div 8$
$64 \div 8$
$64 \div 16$

$21 \div 3$
$42 \div 3$
$84 \div 3$

$24 \div 4$
$24 \div 8$
$48 \div 8$

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Explore 'missing number' / 'empty box' problems

Present children with missing number problems (empty box problems) where they use and apply their knowledge of multiplication facts.

$$\square \div 3 = \square$$

$$\square \div 5 = \square$$

Explore remainders

Present children with missing number problems (empty box problems), where they use and apply their knowledge of multiplication facts.

$$\square \div 10 = \square \text{ r } \square$$

$$\square \div 5 = \square \text{ r } \square$$

$$\square \div 4 = \square \text{ r } 3$$

Ask questions like:

What is the largest possible remainder when the divisor is 5?

How do you know?

If I want the remainder to be 1, what could the dividend be?

How do you know?

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Exploring remainders

Year 3 and 4 (taken from NNS (1999) pages 51 & 56)

Year 3

Give a **whole-number remainder** when one number is divided by another. For example:

16 ÷ 3 is 5 remainder 1

75 ÷ 10 is 7 remainder 5

Respond to oral or written questions such as finding how many are left or how much is left when you:

Share 18 between 5

Divide 25 by 10

Cut lengths of 10cm from 81 cm of tape

Year 4

Give a remainder as a **whole-number**. For example:

41 ÷ 4 is 10 remainder 1

72 ÷ 5 is 14 remainder 2

768 ÷ 100 is 7 remainder 68

There are 64 children in year 5

How many teams of 6 children can be made?

How many children will be left over?

Divide a whole number of pounds by 2, 4, 5 or 10. For example:

Four children collected £19 for charity

They each collected the same amount.

How much did each child collect?

Rounding up or down after division

Make sensible decisions about rounding down or up after division, depending on the context of the problem.

Make sensible decisions about rounding down or up after division, depending on the **context** of the problem.

Examples of rounding down:

I have £46. Tickets cost £5 each.

I can only buy 9 tickets.

I have 46 cakes. One box holds 5 cakes.

I could fill only 9 boxes of cakes.

Examples of rounding up:

I have 46 cakes. One box holds 5 cakes

I will need 10 boxes to hold all 46 cakes.

There are 46 children. A table seats 5.

10 tables are needed to seat all the children.

Make sensible decisions about rounding up or down after division, depending on the context of the problem.

Examples of rounding down:

I have £62. Tickets cost £8 each.

I can only buy 7 tickets.

I have 62 cakes. One box holds 8 cakes.

I could fill only 7 boxes of cakes.

Examples of rounding up:

I have 62 cakes. One box holds 8 cakes

I will need 8 boxes to hold all 62 cakes.

There are 62 people. There are 8 seats in a row.

8 rows of seats are needed to seat everyone.

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Challenge beliefs

(Newstead et al p.7)

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

Challenge common beliefs that:

"division makes smaller"

"the quotient is always smaller than the dividend"

"the divisor must be a whole number"

"the divisor is always smaller than the dividend"

(Newstead et al p.7)

Use questions like:

$4 \div \frac{1}{2}$

$6 \div 1$

$6 \div 12$

Newstead, K., Anghileri, J., Whitebread, D. (n.d.). Language and strategies in children's solutions of division problems. Homerton College: Cambridge.

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Exploring fact boxes

Create a fact box / ‘partial multiple table’ for 36 (Thompson, 2005)

977 ÷ 36

Children need to develop their confidence in multiples of two-digit numbers. They use the basic skills of doubling and halving and multiplying by 10 to create a fact box or ‘partial multiple table’ (Thompson, 2012).

“Three, six or thirty 36s can be found by addition and if the dividend is a larger number, the table can easily be extended”. (Thompson, 2012 p.46).

A fact box - (partial multiples of the divisor)

1 ⟶ 36

2 ⟶ 72

4 ⟶ 144

10 ⟶ 360

5 ⟶ 180

20 ⟶ 720

The fact box should be used get a ‘feel’ for the size of the answer - (estimation) through addition.

Adding twenty multiples and five multiples of 36 will give a total of 900

Adding twenty multiples and ten multiples of 36 will give a total of 1040

Therefore the quotient will be more than 25 but less than 30.

- Use pre-prepared fact boxes to explore what can be calculated using the facts.
- Create fact boxes from a given divisor. Identify questions that you can answer using your fact box.

Spend time exploring fact boxes - developing children’s ‘number sense’ and making the links to known facts (times tables), addition, subtraction and problem solving skills prior to introducing chunking on a number line.

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Additive chunking / 'Chunking up'

"The challenge in division is to keep track of 3 different sets of working numbers simultaneously: the chunks needed to make the calculation efficient, the number of groups i.e. the quotient and finally how much of the dividend has been used/is left. All 3 sets of numbers can be recorded simultaneously on an empty number line, the chunks taking the form of 'jumps'." (Bradford, 2011)

Moving towards efficiency

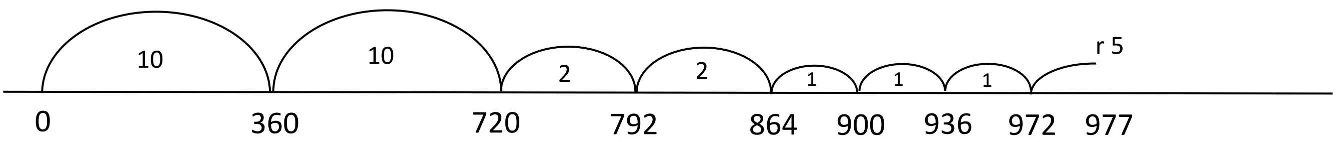
Jumps are representative

977 ÷ 36

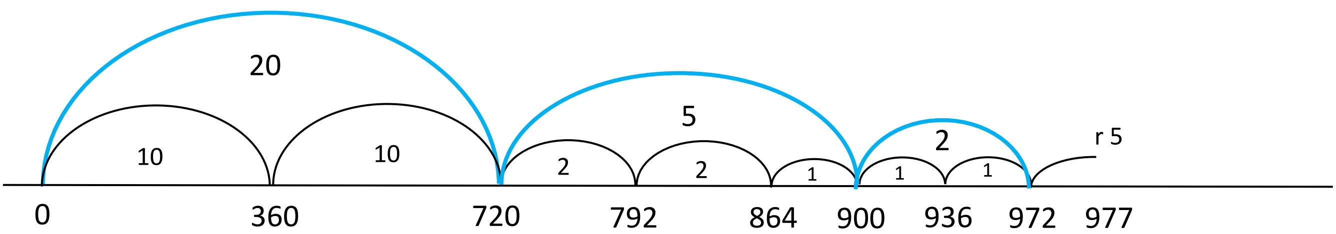
A fact box ('partial multiple table')

1 →	36
2 →	72
4 →	144
10 →	360
5 →	180
20 →	720

Bank
10
10
2
2
1
1
r 5



Children annotate their chunking using a coloured pencil, to show more efficient chunking



977 ÷ 36 = 27 r 5

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Using this fact box —how would you calculate the following?

Moving towards efficiency

977 ÷ 36

A fact box (‘partial multiple table’)

1 ⟶ 36

2 ⟶ 72

4 ⟶ 144

10 ⟶ 360

5 ⟶ 180

20 ⟶ 720

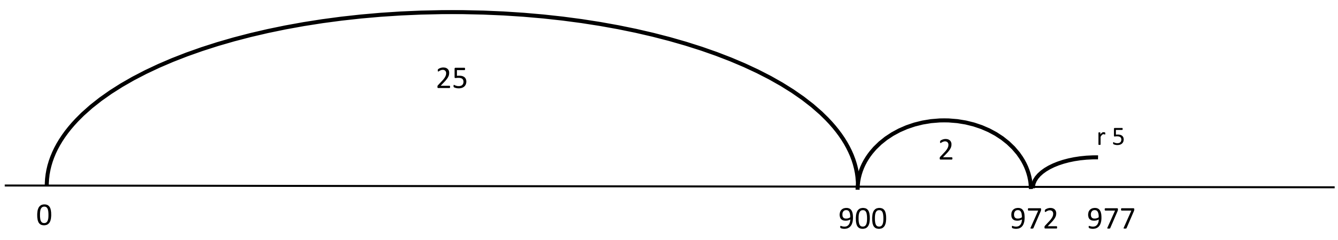
Bank

20

5

2

r 5



977 ÷ 36 = 27 r 5

Using the facts in the fact box

How would you *efficiently* calculate the following?

396 ÷ 36

400 ÷ 36

580 ÷ 36

1000 ÷ 36

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Equivalent Calculations

Replace with an equivalent calculation

100 ÷ 20 is the same as 10 ÷ 2

Write down an equivalent calculation for:

200 ÷ 50

200 ÷ 10

120 ÷ 30

450 ÷ 4.5

The activity may engage pupils in *cognitive conflict*

Cognitive conflict is where learners are faced with discrepancies or conflicting ideas, or a result that differs from their prediction or conflicts with their present understanding. The resolution of this conflict can lead to increased knowledge and understanding. Cognitive conflict in teaching mathematics can therefore be used to promote growth in understanding.

Dividing by 10

Find 6 errors – explain your reasoning

190 ÷ 10 = 19	560 ÷ 10 = 56
206 ÷ 10 = 2 r 6	309 ÷ 10 = 3 r 9
5609 ÷ 10 = 56 r 9	470 ÷ 10 = 47
4097 ÷ 10 = 49 r 7	20.6 ÷ 10 = 2 r 0.6
8.06 ÷ 10 = 8.6	Make up some of your own questions like these

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Years 5 and 6

Sets of questions to develop reasoning about division

$128 \div 4$ $256 \div 8$ $128 \div 8$	$222 \div 3$ $222 \div 6$ $111 \div 3$	$136 \div 8$ $68 \div 4$ $34 \div 2$
$51 \div 3$ $102 \div 3$ $204 \div 3$	$72 \div 9$ $72 \div 18$ $72 \div 3$	$64 \div 4$ $64 \div 8$ $32 \div 4$
$4 \div 2$ $4 \div 1$ $4 \div 4$	$24 \div \frac{1}{2}$ $24 \div 2$ $24 \div 1$	$6 \div 12$ $12 \div 12$ $6 \div 24$

Make links to fractions and simplifying fractions

$$6 \div 12 = \frac{6}{12} = \frac{1}{2}$$

Read: Newstead, K., Anghileri, J., Whitebread, D (n.d. Language and Strategies in Children's Solution of Division Problems. Informal Proceedings 16 – 1 (BSRLM).

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2 chunk and 3 chunk questions – *moving towards efficiency*

Children need experience of using larger multiples of the divisor. This reduces calculation errors and is more efficient.

Ask children to refine their chunking – to become more efficient in creating and applying the facts in their fact box.

Create a fact box or 'partial multiple table' (Thompson 2005) for each division calculation.

Complete these calculations in **3 chunks**	Complete these calculations in **2 chunks**
732 ÷ 6	93 ÷ 7
639 ÷ 4	73 ÷ 6
741 ÷ 8	152 ÷ 4
857 ÷ 7	741 ÷ 8
573 ÷ 34	247 ÷ 11
293 ÷ 17	253 ÷ 17
318 ÷ 13	739 ÷ 12
544 ÷ 32	318 ÷ 13

Ask children to create their own 2 chunk and 3 chunk questions.

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$732 \div 6$	$573 \div 34$
$639 \div 4$	$293 \div 17$
$741 \div 8$	$318 \div 13$
$857 \div 7$	$544 \div 32$

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$93 \div 7$	$247 \div 11$
$73 \div 6$	$253 \div 17$
$152 \div 4$	$739 \div 12$
$741 \div 8$	$318 \div 13$

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A decimal divisor

Use knowledge of table facts and reasoning

57 ÷ 3.4

is the same as

570 ÷ 34

46 ÷ 3.5

is the same as

92 ÷ 7

54 ÷ 2.7

is the same as

540 ÷ 27

70 ÷ 2.5

is the same as

140 ÷ 5

is the same as

280 ÷ 10

75 ÷ 3.75

is the same as

150 ÷ 7.5

is the same as

300 ÷ 15

26 ÷ 1.3

is the same as

260 ÷ 13

A decimal dividend and a decimal divisor

Use knowledge of table facts and reasoning

7.5 ÷ 1.5

is the same as

15 ÷ 3

34.5 ÷ 1.5

is the same as

69 ÷ 3

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Devise your own written methods to solve simple two-digit by one-digit divisions.

When children have some established strategies for division, give them opportunities to "devise their own written methods to solve simple two -digit by one-digit divisions." (Bradford 2011, in Thompson, 2012)

See examples in:

'To Chunk or not to Chunk' by Ian Thompson page 47

and

'Division – forwards or backwards' by Krista Bradford pages 11 and 12

Include division calculations with a range of two-digit divisors

Write questions using these divisors

29, 39, 48, 45, 37, 24, 56

Complete the same calculation in three different ways

739 ÷ 12

- Additive chunking/ 'Chunking up' on a number line
- Own written method
- Long division

Decide which is the most efficient.

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Exploring remainders

Year 5 and 6 (taken from NNS (1999) p.57)

Give a quotient as a **fraction** when dividing by a whole number. For example:

$43 \div 9 = 4\frac{7}{9}$ $90 \div 7 = 12\frac{6}{7}$

Give a quotient as a **decimal fraction**:

- When dividing by 10, 5, 4 or 2, for example:

  51 ÷ 10 = 35.1

  61 ÷ 4 = 15.25

- When dividing by any whole number, for example:

  676÷ 8 = 84.5

  612 ÷ 100 = 6.12

Rounding up or down after division

Make sensible decisions about rounding down or up after division, depending on the context of the problem.

For example, *240 ÷ 52 is 4 remainder 32*, but whether the answer should be *rounded up to 5* or *rounded down to 4* **depends on the context.**

Examples of rounding down:

I have saved £240. A train ticket to Durham is £52.

240 ÷ 52 is 4.61 on my calculator

I can buy only 4 tickets.

I have 240 cakes. One box holds 52 cakes.

I could fill only 4 boxes of cakes.

For example, 1000 ÷ 265 is 3.8, but whether the answer should be rounded up to 4 or rounded down to 3 depends on the context.

Examples of rounding down:

Dad has saved £5000. An air fare to Sydney is £865.

5000 ÷ 865 is 5.780 on my calculator

He can buy 5 tickets.

I have 5 metres of rope. I need lengths of 865 cm.

I can cut off 5 lengths.

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Division

Formal written methods

$$\begin{array}{rl} & 176 \\ 4 & \overline{)\,7\,{}^{3}0\,{}^{2}4} \end{array}$$

$$\begin{array}{rr} & 49.5 \\ 12 & \overline{)\,594.0} \\ & \underline{48} \\ & 1\,{}^{0}\cancel{1}\,{}^{1}4 \\ & \underline{108} \\ & 6.0 \\ & \underline{6.0} \\ & 0 \end{array}$$

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Short division and Long division

Short division

'Divide numbers up to 4 digits by a one-digit number using the formal written method of short division…' (DfE 2103 p.82).

Thompson 2012 states that there "seems to be some confusion about the word 'understanding', in that there is a very great difference between being able to correctly execute the short division algorithm and actually understanding it" (p.46).

Plunkett, 1979; Noss, 1997; Thompson, 1997; Anghileri, 1998 argue "that teaching standardised procedures for calculating encourages 'cognitive passivity' and suspended understanding' as they do not correspond to the way people naturally think about numbers" (Anghileri, p.25).

Anghileri (n.d) states that "Children tend to use algorithms as 'mechanical' procedures. Where they do not understand the procedures, they are unable to reconstruct the processes involved" (p.25).

It is recognised that *The National Curriculum in England (DfE, 2013)*, states in the 'Programme of study statutory requirements' that:

- in year 3 there is a requirement to **progress to** "using formal written methods".
- in year 4 there is a requirement to "add and subtract numbers with up to 4 digits using the formal written methods of columnar addition and subtraction **where appropriate"** and "multiply two-digit and three-digit numbers by a one-digit number using formal written layout."
- in year 5 children are expected to be able to "multiply numbers up to 4 digits by a one or two-digit number using **a** formal written method, including long multiplication for two-digit numbers" and "divide numbers up to 4 digits by a one or two-digit number using the formal written method of short division and interpret remainders appropriately for the context".
- in year 6 children "multiply multi-digit numbers up to 4 digits by a two-digit whole number using the formal written method of long multiplication" and "divide numbers up to 4 digits by a two-digit number using the formal written method of short division where appropriate, interpreting remainders according to the context".

However, we still need to ensure that we look wider and reference research in this area. In 1997 the School Curriculum and Assessment Authority, 1997; Anghileri, 2000b. expressed the view that "place value and rehearsed formal procedures are no longer central as flexibility is called for in matching appropriate strategies to particular problems".

The aim of this policy is to ensure that we respond to the changes and challenges in the new National Curriculum as well as ensuring that this policy is based upon sound pedagogy and subject knowledge and makes reference to a range of research in this area.

Anghileri (n.d), in the final paragraph of her article on British Research on Mental and Written Calculation Methods for Multiplication and Division, succinctly sums up for us the pedagogical approaches that are central to this policy.

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> "Children are expected to interpret problems in a meaningful way, making connections between the conceptual and calculational aspects of mathematics. By focussing on the development of number sense through encouraging mental methods and informal written strategies, children will develop confidence in their own approaches to problem solving and maintain an inclination and enthusiasm for mathe-

In this policy we are not ignoring the 'practise to become fluent in the formal written method … of short division, or the use of 'the formal written method of long division' (DfE, 2013), quite the contrary. We have included these in this policy. "We recognise that there are children in primary schools who use, and can apply these 'efficient methods' flexibly and with understanding and that they also make decisions about 'matching appropriate strategies to particular problems" (Anghileri ,n.d.).

It remains the case though that the language associated with short and long division is what Thompson, (2012) refers to as a 'spiel' (p.46) and it is this spiel that makes it 'very difficult to give meaning to this procedure'.

Examples of this 'spiel' are given on the following pages, as are examples that demonstrate some of the complexities of short division when it is used correctly as an 'efficient method', rather than over– applied to calculations where, if children are indeed using what Anghileri refers to as 'number sense', they would *see*, choose and use other more efficient strategies instead.

Long division HTU ÷ TU

In his article,' United we stand; divided we fall' Ian Thompson (2003) discusses the language and conceptual difficulties with long division;

```
36 )972
    –720  20 x 36
     252
    –252   7 x 36
       0
Answer: 27
```

Example 5

```
36 )972
    –72   20 x 36
     252
    –252   7 x 36
       0
Answer: 27
```

Example 6

"I conceded that the algorithm shown in Example 6 might make sense to some children, but I have yet to be convinced that there can be a smooth transition from the language and reasoning associated with chunking, which is transparent and supportive of children's thinking, to the language of the standard algorithm which is not.

I have argued elsewhere (1999, 2002) that the *quantity* aspect of place value (where 67 is seen as 60 and 7) develops before the more sophisticated *column* aspect (where 6 is in the tens column and 7 in the ones column). The language of chunking clearly involves the quantity aspect, complementing the steps in the procedure, whereas the standard algorithm involves the column aspect as well as a good understanding of the principle of *exchanging*.

So my second question is: Is it worth the effort and the potential risk of confusion involved in teaching the algorithm in Example 6 for the sake of a layout which, in this particular case, has only one zero fewer?" (Thompson, 2003 p.22).

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Short division

$$8\overline{)104}$$

"...the spiel goes something like this:

8 into 1 doesn't go [although it actually does if the one has its real value of 100]...

8 into 10 goes one remainder 2... [8 into 10 what?]...

Carry the 2...

8 into 24 goes 3"

"It is very difficult to give meaning to this procedure - it just has to be learned - and forgotten, as much of the research in the 1980s showed us." (Thompson 2012 p.46)

No exchange

$$\begin{array}{r} 121 \\ 4\overline{)484} \end{array}$$

Why would we do this calculation using short division?

Could we be teaching a misconception?

If children have reached the stage where they are being presented with a calculation in the format of 'short division', *then it must be the case* that they have all the skills, knowledge and understanding necessary to carry out these calculations mentally!

$$2\overline{)428} \qquad 3\overline{)639}$$

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Short division

Continue to use a fact box where appropriate

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

With one-digit exchanges

$$\begin{array}{r}218\\3\,\overline{)\,6\,5\,{}^{2}4}\end{array}$$

$$\begin{array}{r}194\\3\,\overline{)\,5\,{}^{2}8\,{}^{1}2}\end{array}$$

$$\begin{array}{r}145\text{r}1\\5\,\overline{)\,7\,{}^{2}2\,{}^{2}6}\end{array}$$

Placing the quotient

This is incorrect

Children need to learn to place the quotient.

$$\begin{array}{r}044\\6\,\overline{)\,2\,{}^{2}6\,{}^{2}4}\end{array}$$

$$\begin{array}{r}42\\7\,\overline{)\,29\,{}^{1}4}\end{array}$$

Zero in the quotient

$$\begin{array}{r}204\\4\,\overline{)\,81\,{}^{1}6}\end{array}$$

With exchange

Zero in the quotient

$$\begin{array}{r}306\\3\,\overline{)\,91\,{}^{1}8}\end{array}$$

With exchange

Zero in the quotient

$$\begin{array}{r}701\\8\,\overline{)\,5608}\end{array}$$

Placing the quotient

No exchange

Zero in the quotient

Zero in the dividend

Zero in the dividend

$$\begin{array}{r}176\\4\,\overline{)\,7\,{}^{3}0\,{}^{2}4}\end{array}$$

$$\begin{array}{r}150\text{r}4\\6\,\overline{)\,9\,{}^{3}05}\end{array}$$

Zero in the quotient

Zero in the dividend

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Short division

Continue to use a fact box to get a *sense of the size of the answer*

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

Short division - with a two digit divisor

With two-digit divisor - applying known times table facts

Quotient expressed as a fraction

$594 \div 12 = 49\cancel{\tfrac{6}{12}}\ \tfrac{1}{2}$

$$\begin{array}{r|l} & 4\ 9\ r\ 6 \\ \hline 12 & 5\ 9\ {}^{11}4 \end{array}$$

Placing the quotient

With two-digit exchange

With two-digit divisor - applying known times table facts

Quotient expressed as a remainder

$389 \div 11 = 35\ r\ 4$

$$\begin{array}{r|l} & 3\ 5\ r\ 4 \\ \hline 11 & 3\ 8\ {}^{5}9 \end{array}$$

Placing the quotient

With one-digit exchange

Short division - with a decimal dividend and decimal quotient

Decimal dividend

$$\begin{array}{r|l} & 1\ 2\ .\ 5 \\ \hline 7 & 8\ {}^{1}7\ .\ {}^{3}5 \end{array}$$

With exchange

Decimal dividend

Placing the quotient

Quotient has 2 decimal places

$$\begin{array}{r|l} & 1\ 6\ .\ 1\ 5 \\ \hline 8 & 1\ 2\ {}^{4}9\ .\ {}^{1}2\ {}^{4}0 \end{array}$$

Placing the quotient

With exchange

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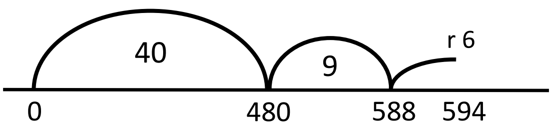
Moving from informal to formal written methods. Look at informal and formal written methods for the same calculation and ask children to notice what is the same and what is different.

A fact box

1	→	12
2	→	24
4	→	48
10	→	120
5	→	60
20	→	240
40	→	480



$$594 \div 12 = 49 \text{ r } 6$$

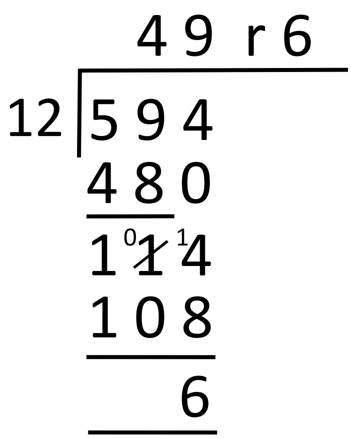
A fact box

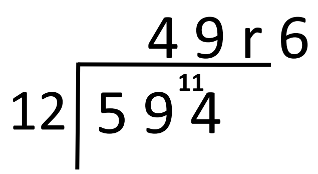
1	→	12
2	→	24
4	→	48
10	→	120
5	→	60
20	→	240
40	→	480

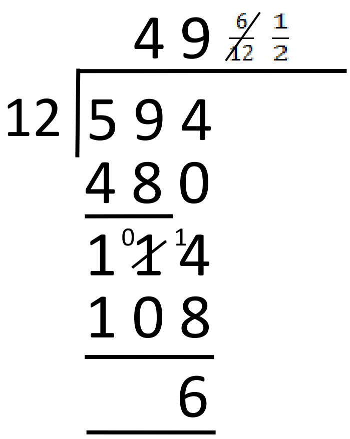
Informal with jottings

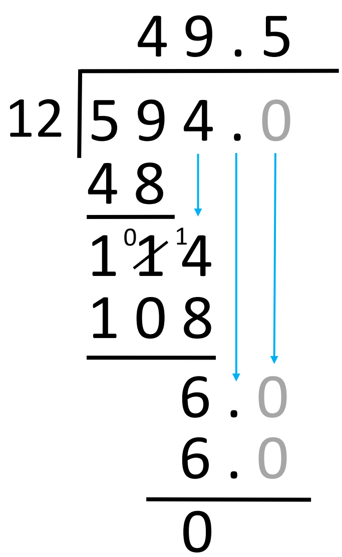
5 x 12 = 60 so 50 x 12 = 600
600 this is 6 more than 594 so
the quotient must be 49 r 6

50 x 12 = 600
This is 6 more than 594 so
the quotient must be 49 r 6









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Long division - *continue to use a fact box where appropriate*

With two-digit divisor

Quotient expressed as a whole number with a remainder, fraction, or decimal depending on context - refer to information on remainders.

```
      2 3 r 7
15 | 3 5 2
     3 0 0
     ⁴5̸ ¹2
       4 5
         7
```

With two-digit divisor

Quotient expressed as a whole number with a remainder, fraction or decimal depending on context - refer to information on remainders.

$$26\,\cancel{\tfrac{28}{35}}\;\tfrac{4}{5}$$

```
35 | 9 3 8
     7 0 0
     2 3 8
     2 1 0
       2 8
```

With two-digit divisor

Quotient expressed as a whole number and a decimal

```
        2 6 . 8
35 | 9 3 8 . 0
     7 0
     2 3 8
     2 1 0
       2 8 . 0
       2 8 . 0
```

When you find that the quotient will not be an integer (whole number) - put in the decimal point and the zero so that the quotient can be calculated and expressed as a decimal.

Bring the 8 down.

Bring the decimal point down.

Bring the zero down.

Read this as 280 (tenths) and not as 28.

So we ask ourselves how many 35s there are in 280

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Appendix

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$17 \div 2$	$72 \div 5$
$23 \div 3$	$47 \div 5$
$11 \div 2$	$93 \div 10$
$23 \div 2$	$46 \div 5$
$91 \div 2$	$84 \div 10$
$19 \div 8$	$71 \div 5$

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$31 \div 8$	$56 \div 10$
$39 \div 8$	$56 \div 5$
$15 \div 8$	$27 \div 4$
$36 \div 8$	$83 \div 4$
$20 \div 8$	$22 \div 4$
$25 \div 3$	$54 \div 4$

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$80 \div 5$	$54 \div 3$
$55 \div 5$	$27 \div 3$
$70 \div 5$	$51 \div 3$
$32 \div 8$	$98 \div 2$
$80 \div 8$	$54 \div 2$
$64 \div 8$	$78 \div 2$

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$24 \div 8$	$36 \div 2$
$90 \div 10$	$88 \div 4$
$70 \div 10$	$24 \div 4$
$120 \div 10$	$36 \div 4$
$60 \div 10$	$28 \div 5$
$24 \div 3$	$95 \div 5$

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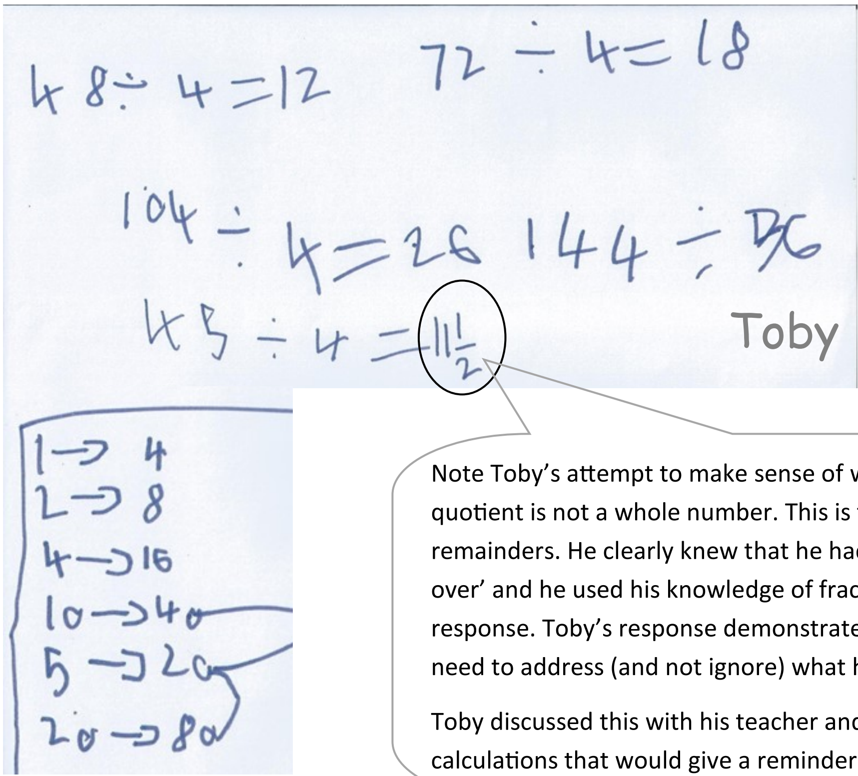
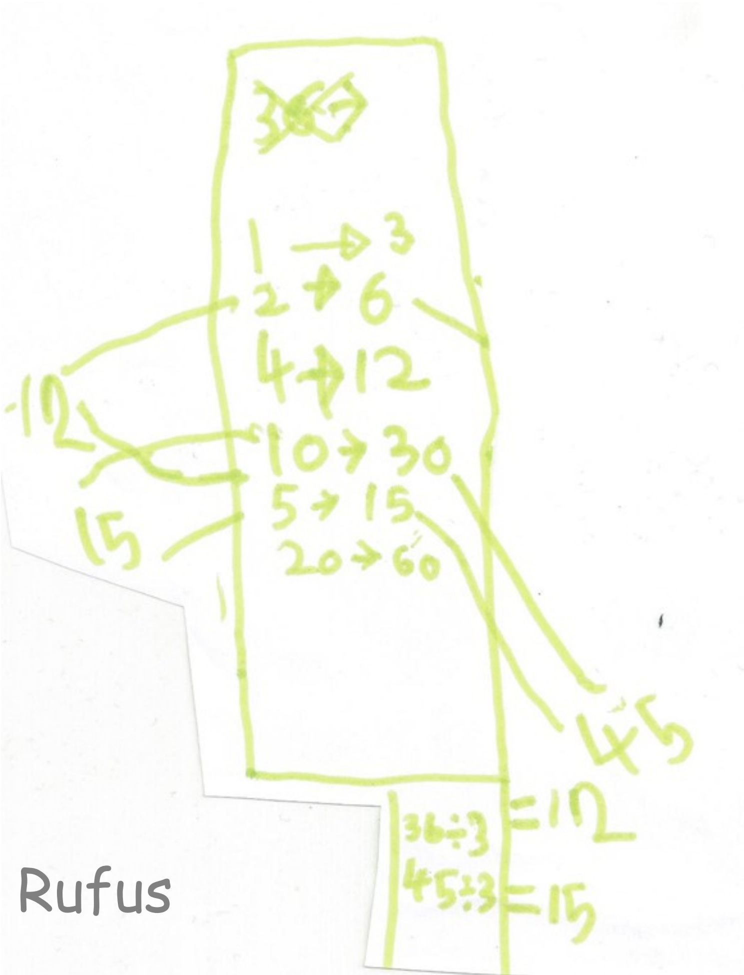
Year 2 children using a fact boxes to calculate the quotient



Note Toby's attempt to make sense of what happens when the quotient is not a whole number. This is the first time he had met remainders. He clearly knew that he had to address the 'bit left over' and he used his knowledge of fractions to offer an plausible response. Toby's response demonstrates that he understands the need to address (and not ignore) what he had discovered.

Toby discussed this with his teacher and went on the work on more calculations that would give a reminder.



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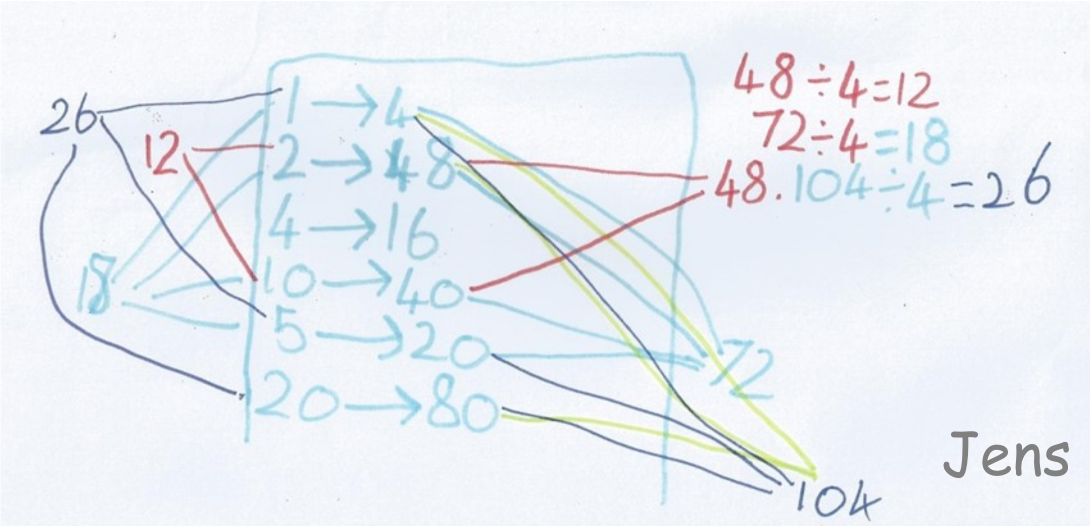
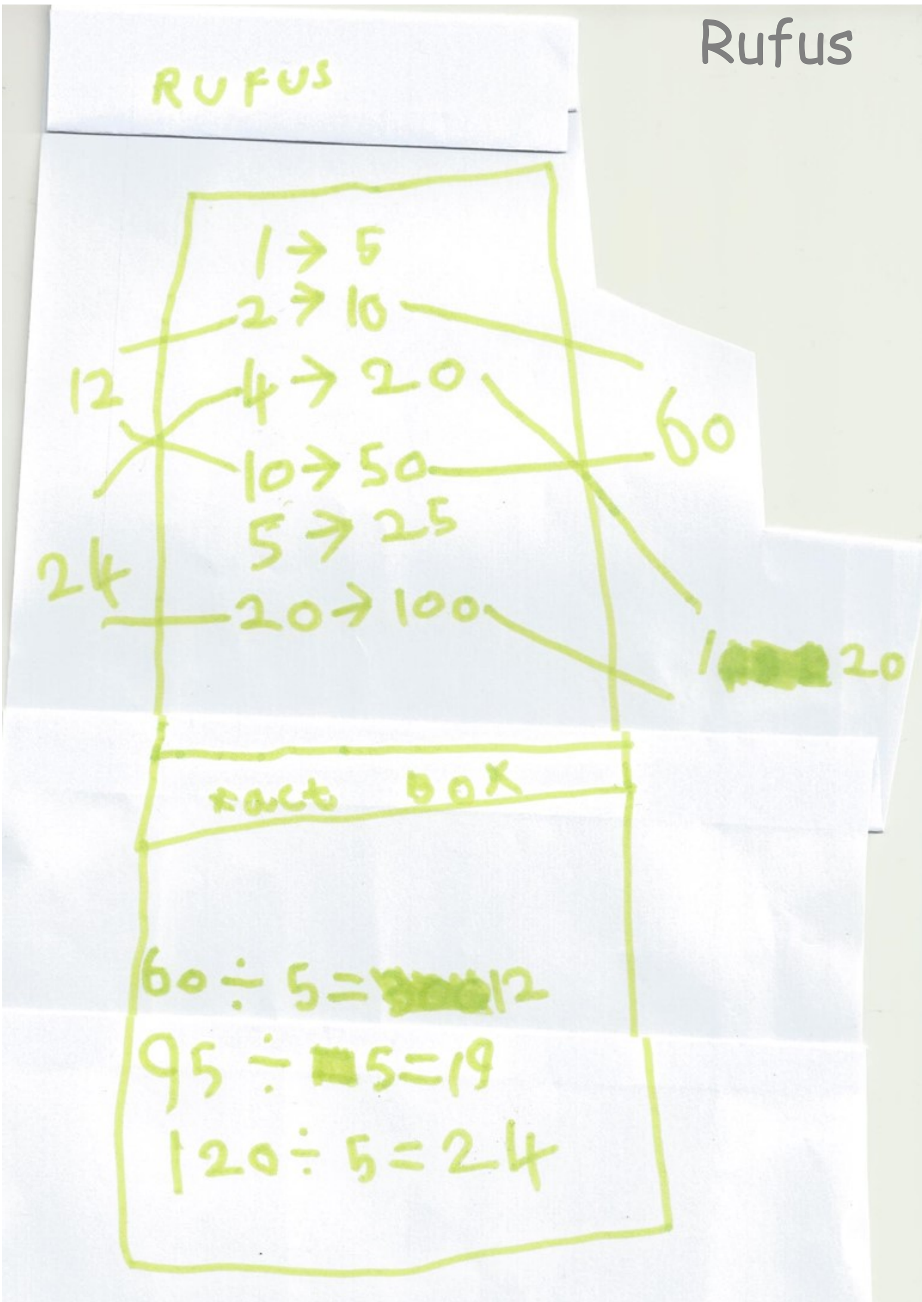
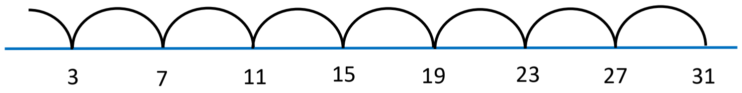
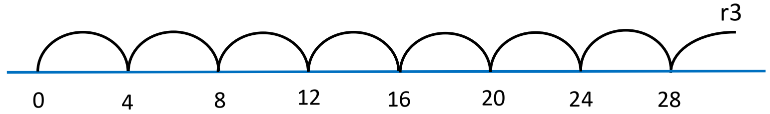
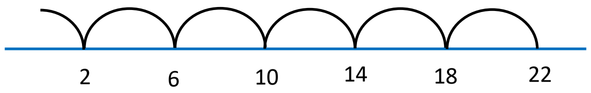
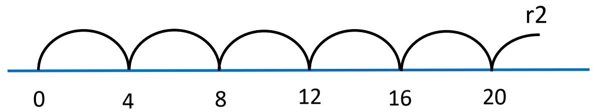
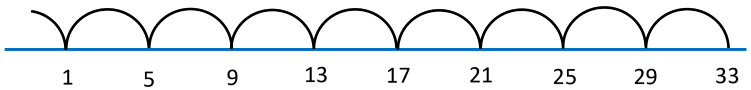
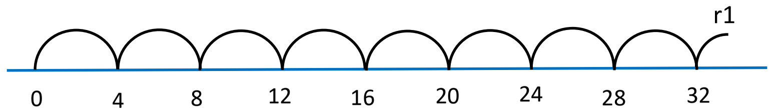

48÷4=12
72÷4=18
48.104÷4=26
26
12
18
1→4
2→8
4→16
10→40
5→20
20→80
72
104
Jens


Rufus
RUFUS
1→5
2→10
4→20
10→50
5→25
20→100
12
24
60
120
fact box
60÷5=12
95÷5=19
120÷5=24

The rationale for additive chunking,' chunking up'

Calculation	Subtractive chunking	Commentary	Additive chunking/ 'chinking up'	Commentary
31 ÷ 4		Working backwards gives an 'unfamiliar' sequence of numbers. Where the dividend changes but the divisor remains the same, the sequence of numbers is different each time. This does not allow children to utilise and apply their growing knowledge of times table facts.		Working forwards, 'counting up' gives a familiar sequence - the sequence of the four times table. It reinforces children's re-call and appli-cation of the four times ta-ble. The 'familiar sequence' is easier to check for errors.
22 ÷ 4				
33 ÷ 4				

Reference: Thompson, I. (2012) To chunk or not to chunk? *Mathematics Teaching 227*, Journal of Association of Teachers of Mathematics.

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The rationale for additive chunking,' chunking up'

Calculation	Subtractive chunking	What do you notice?	Additive chunking/ 'chunking up'	What do you notice?
31 ÷ 4				
22 ÷ 4				
33 ÷ 4				

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Fact box

product

remainder

$$12 \div 4 = 3$$

dividend ÷ divisor = quotient

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